The Riemann Problem for Reynolds-Stress-Transport in RANS and VLES

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1 Introduction

Recent work on the numerical computation of the Navier-Stokes equations with Reynolds-stress model (RSM) 7-equation turbulence closures, both in a Reynolds-averaged (RSM–RANS) framework, or in continuous RANS-to-DNS RSM–VLES approaches [1], has produced numerical methods which allow the evaluation and improvement of these advanced closures. However, especially when RSM–VLES approaches are concerned, it is important to use low-diffusion [2] high-order schemes.

The use of low-diffusion approximate Riemann solvers (ARSs) using a passive-scalar approach for the Reynolds-stresses [3, 4] fails even for simple subsonic flows (Fig. 1). In a recent work [5] this was related to the incorrect treatment of the contact discontinuity. In a classic low-diffusion solver, the massflux has no dissipation for a stationary contact discontinuity [2] and treating the Reynolds-stresses as passive scalars yields the incorrect condition that pressure is continuous across a contact discontinuity, whereas the correct condition is that \( \bar{p} + \bar{\rho} r_n n \) should be continuous. One solution is using a hybrid scheme, with a dissipative massflux for the Reynolds-stress transport equations [5]. This solution can be used with a
variety of ARSs for the meanflow equations, and yields robust schemes applicable to complex flows [5].

In the present work we examine the Riemann problem for RST, and develop an HLLC–RSM flux for the coupled system of equations.

\[
\begin{align*}
\text{HLLC--}\text{conv.} & \\
\text{HLLC--}m & = 400 \\
\text{HLLC--}m & = 800 \\
\text{HLLC--}m & = 1200 \\
\text{HLLC--}m & = 2800
\end{align*}
\]

experiment (Acharya, 1977; \( \text{M}_e = 0.52; \text{Re}_\theta = 12000 \))

experiment (Acharya, 1977; \( \text{M}_e = 0.69; \text{Re}_\theta = 33000 \))

Fig. 1. Mean-mass-flux \( \overline{\rho u} \), logarithmic law \( u^+ \), and Reynolds-stresses for near-zero-pressure-gradient boundary-layer flow, using the HLLC ARS with the passive-scalar approach for the Reynolds-stresses (comparison with measurements of Acharya [6] at \( M_e = 0.22; \text{Re}_\theta = 21000 \), and at \( M_e = 0.6; \text{Re}_\theta = 33000 \)).

2 Reynolds-Stress Transport

2.1 The Complete Set of Equations

The equations are separated into a convective part (time-derivatives and first-derivatives), a diffusive part \( D \) (second-derivatives), and source-terms \( S \) (which do not contain derivatives, or are modelled terms such as the rapid part of redistribution).

\[
\begin{align*}
\frac{\partial}{\partial t} & \begin{bmatrix}
\rho \overline{u}_i \\
\rho \overline{h}_i - \rho \\
\rho v_{ij} \\
\rho \varepsilon_w
\end{bmatrix}
+ \frac{\partial}{\partial x_i} & \begin{bmatrix}
\rho \overline{u}_i \overline{u}_j + \rho \delta_{ij} + \rho v_{ij} \\
\rho \overline{u}_i \overline{u}_j + \rho v_{ij} \overline{u}_k \\
\rho v_{ij} \overline{u}_j \\
\rho \varepsilon_w \overline{u}_j
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
- P_{ij}
\end{bmatrix}
= D + S 
\end{align*}
\]  

(1)

where the red terms correspond to the coupling of the Reynolds-stress with the meanflow equations (through the convective fluxes), while the green terms correspond to nonconservative terms coming form \( -P_{ij} = \rho v_{ij} \partial_{x_i} \overline{u}_j + \rho v_{ij} \partial_{x_j} \overline{u}_i \) (exact terms).
2.2 Eigenvalues and Eigenvectors

Retaining the production terms in the RST, the system can be recast in matrix form

$$\frac{\partial v}{\partial t} + A \frac{\partial v}{\partial x} = 0$$

where $A \in \mathbb{R}^{12 \times 12}$ are nonstrictly hyperbolic matrices (12 real eigenvalues with multiplicity) which are not Jacobians of a flux-vector (nonconservative system). Considering, without loss of generality $A$

$$\begin{pmatrix}
\bar{\rho} & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\rho} & \bar{\rho} & 0 & 0 & 0 & 0 & \sqrt{a^2 + 3r_{xx}} & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\rho} & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\bar{\rho} \\
\end{pmatrix}$$

it is straightforward to show that the eigenvalues are

$$\lambda_L = \bar{u} - \sqrt{a^2 + 3r_{xx}}$$ (4a)
$$\lambda_{L+} = \bar{u} - \sqrt{r_{xx}}$$ (4b)
$$\lambda_+ = \bar{u} +$$ (4c)
$$\lambda_{B+} = \bar{u} + \sqrt{r_{xx}}$$ (4d)
$$\lambda_L = \bar{u} + \sqrt{a^2 + 3r_{xx}}$$ (4e)

The analysis of a similar reduced problem by Berthon et al. [7] reveals that the Riemann problem solution for this system is composed of 2 genuinely nonlinear (GNL) waves and 3 linearly degenerate contact discontinuities (Fig. 2), thus containing 6 instead of 4 possible states for the construction of an HLLC-type flux.

2.3 Approximate Jump Relations

The nonconservative products are treated by connecting states across the discontinuity with a linear path [7]. In that case, the jump relations across a discontinuity with speed $s$, separating states 1 and 2, read
2.4 Approximate Jump Relations for $\lambda \neq \hat{u}$

A straightforward solution can be obtained for $\Delta [\rho r_{xx}]$ from (Eq. 5e), for $\lambda \neq \hat{u}$,

$$\Delta [\rho r_{xx}] = \frac{3\hat{p}_1 r_{xx} \Delta \hat{u}}{S - \hat{u}_1 - 2\Delta \hat{u}} \quad (6a)$$

Also, obviously (Eq. 5k) $\epsilon^*$ is a passive scalar for this system of equations.
Approximate Jump Relations for $\lambda = \tilde{u} \pm \sqrt{\tilde{r}_{xx}}$

In that case all the other approximate jump-relations (Eqs. 5) can be expressed as functions of $\Delta \tilde{u}$

$$\Delta \tilde{u} = \frac{2\tilde{\rho}_1 \tilde{r}_{xy_1} \Delta \tilde{u}}{(S - \tilde{u}_1 - \frac{3}{2} \Delta \tilde{u})(S - \tilde{u}_1)\tilde{\rho}_1 - (\tilde{\rho}_1 \tilde{r}_{xx_1} + \frac{1}{2} \Delta \tilde{\rho} \tilde{r}_{xx})}$$  
(7a)

Approximate Jump Relations for $\lambda = \tilde{u} \pm \sqrt{\tilde{r}_{xx}}$

In this case (LD-wave) it is reasonable to assume

$$S = \tilde{u}_1 \pm \sqrt{\tilde{r}_{xx_1}} = \tilde{u}_2 \pm \sqrt{\tilde{r}_{xx_2}}$$  
(8)

Using these relations (Eqs. 8), in conjunction with the density jump relation (Eq. 5a), in the equations for $\Delta \tilde{\rho} \tilde{r}_{xx}$ (Eq. 6a), $\Delta \tilde{u}$ (Eq. ??), and in the $x$-momentum jump-relation (Eq. 5b) yields

$$\tilde{r}_{xx_1} = \tilde{r}_{xx_2}$$  
(9a)
$$\tilde{u}_1 = \tilde{u}_2$$  
(9b)
$$\tilde{\rho}_1 = \tilde{\rho}_2$$  
(9c)
$$\tilde{\rho}_1 \tilde{r}_{xx_1} = \tilde{\rho}_2 \tilde{r}_{xx_2}$$  
(9d)
$$\tilde{p}_1 = \tilde{p}_2$$  
(9e)

$$\pm \tilde{\rho}_1 \Delta \tilde{v} = \Delta [\tilde{\rho} \tilde{r}_{xy}]$$  
(10a)
$$\pm \tilde{\rho}_1 \Delta \tilde{w} = \Delta [\tilde{\rho} \tilde{r}_{zx}]$$  
(10b)

2.5 Approximate Jump Relations for $\lambda = \tilde{u}$

On the contact discontinuity corresponding to the eigenvalue $\lambda = \tilde{u}$, it is reasonable to assume

$$S_* = u_{L*} = u_{R*} = u_*$$  
(11)

as in the case of the HLLC ARS for the Euler equations [4, 8]. Then the approximate jump relations become

$$\tilde{\rho}_{L*} + \tilde{\rho}_{R*} \tilde{r}_{xx_{L*}} = \tilde{\rho}_{R*} + \tilde{\rho}_{R*} \tilde{r}_{xx_{R*}}$$  
(12a)
$$\tilde{\rho}_{L*} \tilde{r}_{xy_{L*}} = \tilde{\rho}_{R*} \tilde{r}_{xy_{R*}}$$  
(12b)
$$\tilde{\rho}_{L*} \tilde{r}_{zx_{L*}} = \tilde{\rho}_{R*} \tilde{r}_{zx_{R*}}$$  
(12c)
$$\tilde{u}_{L*} = \tilde{u}_{R*}$$  
(12d)
$$\tilde{v}_{L*} = \tilde{v}_{R*}$$  
(12e)
$$\tilde{w}_{L*} = \tilde{w}_{R*}$$  
(12f)
2.6 Closure Relations for the HLLC–RSM Flux

Using the the jump-relations across the 2 GNL-waves we can determine the various states in the HLLC–RSM ARS, viz

\[
S_s = \left[ \hat{\rho}_L(S_L - \hat{u}_L)\hat{u}_L - (\hat{\rho} + \hat{\rho}r_{xx})_L \right] - \left[ \hat{\rho}_R(S_R - \hat{u}_R)\hat{u}_R - (\hat{\rho} + \hat{\rho}r_{xx})_R \right] \over \hat{\rho}_L(S_L - \hat{u}_L) - \hat{\rho}_R(S_R - \hat{u}_R) \]  
\tag{13a}

\[
\hat{\rho}_{LL+}r_{xxL} = \hat{\rho}_{L+}r_{xxL} = \frac{3\hat{\rho}_L r_{xxL}(S_s - \hat{u}_L)}{S_L - \hat{u}_L - 2(S_s - \hat{u}_L)} \]  
\tag{13b}

\[
\hat{\rho}_{LL+} = \hat{\rho}_{L+} = (\hat{\rho} + \hat{\rho}r_{xx})_L + (S_s - \hat{u}_L)\hat{\rho}_L(S_L - \hat{u}_L) - \frac{3\hat{\rho}_L r_{xxL}(S_s - \hat{u}_L)}{S_L - \hat{u}_L - 2(S_s - \hat{u}_L)} \]  
\tag{13c}

Obviously for the HLLC–RSM ARS the tangential velocities are not passive scalars. They are continuous across the \( \lambda = \hat{u} \) LD-wave, but not across the \( \lambda = \hat{u} \pm \sqrt{\rho_{x}^2 + 3S_{xx}} \) GNL-waves nor across the \( \lambda = \hat{u} \pm \sqrt{S_{xx}} \) LD-waves. Using the appropriate jump relations for \( \hat{v} \) (Eqs. 12b, 12e, 10b, 7a) it follows that

\[
\hat{u}_{L+} = \hat{u}_{R+} = \frac{\left( \hat{\rho}_{LL+} \sqrt{r_{xxL+}} \hat{v}_{LL+} + \hat{\rho}_{RR+} \sqrt{3S_{xxL+}} \hat{v}_{RR+} + \hat{\rho}_{BR+} \hat{T}_{xyL+} - \hat{\rho}_{LL+} \hat{T}_{xyL+} \right)}{\hat{\rho}_{LL+} \sqrt{r_{xxL+}} + \hat{\rho}_{RR+} \sqrt{S_{xxL+}}} \]  
\tag{14}

with a similar relation for \( \hat{v} \). The above relations completely define the HLLC–RSM flux.

3 Conclusions

Reynolds-stresses transport cannot be accomodated, in low-diffusion (contact-discontinuity resolving) ARSs, by simply using the passive scalar approach. In the present work we developed an HLLC–RSM ARS for RST.

References